# Fefferman's Counterexample for the Ball Multiplier

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**Notations**

- We use ˆ for the Fourier transform and $^\vee$ for the inverse.
- $\mathcal{S}(\mathbb{R}^d)$ means the Schwartz space.
- $X \lesssim Y$ means $X \leq CY$ for some constant $C$ which is not important.

# 1 Introduction

Given a bounded function $\mathfrak{m} : \mathbb{R}^d \to \mathbb{C}$, we define the operator $\mathfrak{m}(D)$ as follows:

$$\mathfrak{m}(D)f := (\mathfrak{m}\hat{f})^\vee, \ f \in \mathcal{S}(\mathbb{R}^d).$$

The effect of $\mathfrak{m}(D)$ is to multiply $\mathfrak{m}$ to $\hat{f}$, so we call it a multiplier. Note that $\mathfrak{m}(D)$ is $L^2$ bounded by Plancherel. We are interested in its behavior on other $L^p$ spaces for $1 < p < \infty$.

Below we focus on the $d = 2$ case, but many discussions extend to general $d$.

A natural choice of $\mathfrak{m}$ is the characteristic function of certain subsets in $\mathbb{R}^2$.

**Example.** *Let $\mathfrak{m} := 1_A, A \subset \mathbb{R}^2$. We can choose $A$ to be a half plane, a polygon, a pentagram, or a ball.*

It turns out the multipliers associated with half planes, polygons, and pentagrams are not so different. Using the basic singular integral theory, we can prove:

**Theorem 1.** *Let $\mathfrak{m}$ be the characteristic function of a half plane, a polygon, or a pentagram. Then for any $1 < p < \infty$, $\mathfrak{m}(D)$ is an $L^p$ bounded operator.*

However, the ball is very different from others. Its curved boundary provides too many directions which allow for some Kakeya type configuration and give unboundedness.

**Theorem 2** (Fefferman [2])**.** *Let $\mathfrak{m}$ be the characteristic function of a ball. Then for any $p \neq 2$, $\mathfrak{m}(D)$ is not an $L^p$ bounded operator.*

Our main goal is to present Fefferman's proof of Theorem 2. By translation and dilation in the frequency space, it suffices to consider the unit ball. Fix $B := \{\xi \in \mathbb{R}^2 : |\xi| < 1\}$ from now on. We can also assume $p > 2$ by duality. The goal is to construct $f$ such that

$$\|f\|_p^p \text{ is small,} \tag{1}$$
$$\|1_B(D)f\|_p^p \text{ is large.} \tag{2}$$

We will first explain the main ideas and then sketch the rigorous construction.

# 2 The Uncertainty Principle

To study $1_B(D)$, we need to understand how the Fourier support of one function determines its behavior in the physical side.

Let $A \subset \mathbb{R}^2$ be a bounded set near 0. Assume $f$ has Fourier support inside $A$. Let $x_0, x \in \mathbb{R}^2$. We compare $f(x)$ and $f(x_0)$:

$$\begin{aligned} f(x) - f(x_0) &= \int_{\mathbb{R}^2} e^{2\pi i x\cdot\xi}\hat{f}(\xi)d\xi - \int_{\mathbb{R}^2} e^{2\pi i x_0\cdot\xi}\hat{f}(\xi)d\xi \\ &= \int_A (e^{2\pi i(x-x_0)\cdot\xi} - 1)e^{2\pi i x_0\cdot\xi}\hat{f}(\xi)d\xi. \end{aligned}$$

Intuitively, if $x_0, x$ satisfy

$$|(x - x_0)\cdot\xi| \le \frac{1}{100},\ \forall \xi \in A,$$

then $|f(x) - f(x_0)|$ should be small. This motivates us to define the dual set:

$$A^* := \{y \in \mathbb{R}^2 : |y\cdot\xi| \le c \text{ for any } \xi \in A\},$$

where $c$ is a fixed small constant. (The precise value of $c$ is not important.) The above discussion suggests

$$f(x) \approx f(x_0),\ \forall x \in x_0 + A^*.$$

Such intuition is particularly useful when $A$ is convex. For a non-convex set $A$, let $A_0 := A$ and

$$A_1 := \bigcup_{\lambda\in[0,1]} \big((1-\lambda)A_0 + \lambda A_0\big).$$

It's not hard to see $|y\cdot\xi| \le c$ holds on the larger region $A_1$:

$$|y\cdot\xi| \le c,\ \forall \xi \in A_0 \Longrightarrow |y\cdot\xi| \le c,\ \forall \xi \in A_1.$$

Then define $A_2, A_3, \cdots$ recursively. We get that $|y\cdot\xi| \le c$ holds on all $A_j$ by induction. Our construction implies the union

$$A_\infty := \bigcup_j A_j$$

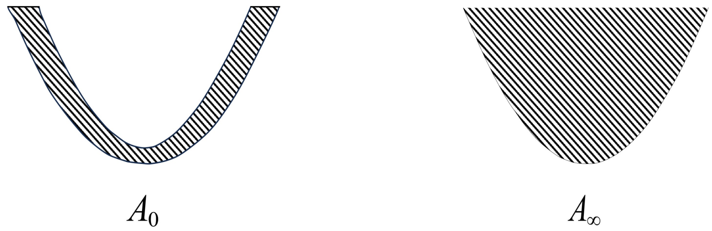


Figure 1: Convex Hull

is the convex hull of $A$. (Equivalently, $A_\infty$ consists of all finite convex combinations of elements in $A$.) And the above argument gives:

$$A^* = A_\infty^*.$$

**Example.** *See Figure 1. $A_0$ is a small neighborhood of the truncated parabola, and $A_\infty = A_1$ is a much larger region.*

When applying the above intuition to a non-convex set $A$, the conclusion will be the same if $\hat{f}$ is supported in a much larger set $A_\infty$. In this case, the intuition becomes inefficient. Thus we only apply it for convex sets. A typical choice of convex sets in $\mathbb{R}^2$ is rectangles. For a rectangle centered at 0, its dual set is essentially the rectangle with the same center and direction, but reciprocal side length. Denote rectangles in the frequency space and the physical space by $\Theta$ and $\mathcal{R}$ respectively.

If $\hat{f}$ is supported in some $\Theta$ not close to 0, we can pick $\xi_0 \in \Theta$ and define $g$ via

$$\hat{g}(\xi) = \hat{f}(\xi + \xi_0).$$

Then $\hat{g}$ is supported in $\Theta_0 := \Theta - \xi_0$ near the origin. Applying our intuition to $g$ gives

$$f(x) \approx g(x_0)e^{2\pi i \xi_0 x}, \ \forall x \in x_0 + \Theta_0^*.$$

Moreover, if $\hat{f}$ is very smooth, integrating by parts several times will give the rapid decay of $|g(x_0)|$ as $|x_0|$ increases. We then expect $f$ is negligible outside $\Theta_0^*$.

We summarize the above discussion as follows:

**Intuition 1** (The Uncertainty Principle)**.** *Assume $\hat{f}$ is supported in a rectangle $\Theta$ with center $\xi_0$. Then*

$$f(x) \approx \sum_{\mathcal{R} \| \Theta_0^*} c_{\mathcal{R}} e^{2\pi i \xi_0 x} 1_{\mathcal{R}}(x),$$

*where $\mathcal{R} \parallel \Theta_0^*$ ranges a tiling of $\mathbb{R}^2$. Assume additionally that $\hat{f}$ is very smooth. Then*

$$f(x) \approx c_{\Theta_0^*} e^{2\pi i \xi_0 x} 1_{\Theta_0^*}(x).$$

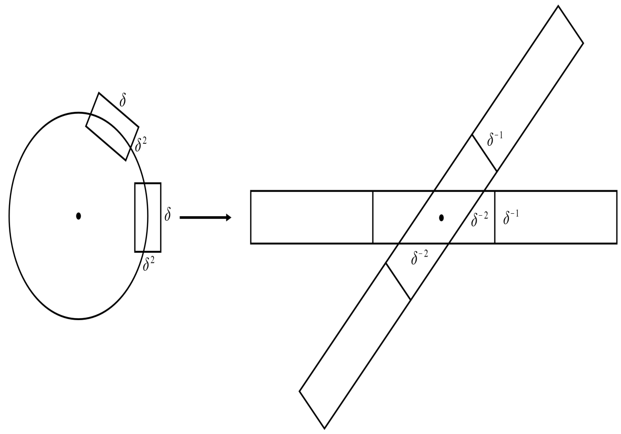


Figure 2: Wave Packets

# 3 Wave Packets for the Circle

Recall we want to construct $f \in \mathcal{S}(\mathbb{R}^2)$ such that (1) and (2) hold. Since the Fourier transform is a bijection, we instead consider what $\hat{f}$ should be like.

An immediate observation is that all interesting actions happen near the boundary $\mathbb{S}^1 := \{\xi : |\xi| = 1\}$. In fact, let $\rho$ be a smooth version of $1_{|\xi|<1-\frac{1}{100}}$. We can write

$$1_B = \rho 1_B + (1-\rho)1_B = \rho + (1-\rho)1_B.$$

Because $\rho(D)$ is just convolution with the smooth function $\check{\rho}$, boundedness of $1_B(D)$ depends on the latter term. This suggests that for our counterexample $\hat{f}$, only the part near boundary matters.

Thus we focus on constructing $\hat{f}$ supported near the boundary. We want to analyze how our counterexample behaves under $1_B(D)$ via the uncertainty principle. But the involved region, a thin annulus near $\mathbb{S}^1$, is far from being convex. To settle this, we fix a small parameter $\delta > 0$ and proceed as follows:

- Partition the annulus of width $\delta^2$ into $\delta \times \delta^2$ rectangles (i.e. small convex regions).
- Construct the corresponding part of $\hat{f}$ associated to each rectangle separately.
- Arrange these parts suitably so that the sum of them gives a counterexample.

See Figure 2. If we put a bump function $\varphi$ on one $\delta \times \delta^2$ rectangle $\Theta$ centered at $\xi_\Theta$, then

$$\check{\varphi}(x) \approx e^{2\pi i \xi_\Theta x} 1_{\Theta_0^*}.$$

Here $\Theta_0^*$ is essentially a $\delta^{-1} \times \delta^{-2}$ rectangle centered at 0. After multiplying $1_B$, the part outside $B$ is eliminated. And we get a less smooth function supported in a smaller "rectangle". Thus the norm of its inverse Fourier transform should be constant in a larger region and decay slower:

$$(1_B\varphi)^\vee(x) \approx e^{2\pi i \xi_\Theta x}, \ \forall x \in \widetilde{\Theta_0^*},$$

where $\widetilde{\Theta_0^*}$ is obtained from $\Theta_0^*$ by extending its longer side 20 times (say). Though $(1_B\varphi)^\vee$ decays slower outside $\widetilde{\Theta_0^*}$, we will still neglect the outside part for simplicity. (The tail turns out to be harmless.)

More generally, we can put a modulated bump function on one $\delta \times \delta^2$ rectangle. This makes $\Theta_0^*$ and $\widetilde{\Theta_0^*}$ move in the physical space together (and gives some extra constant).

We summarize the above discussion as follows:

**Intuition 2** (Effect of the Ball Multiplier)**.** *Consider a modulated bump function $\varphi$ on one $\delta \times \delta^2$ rectangle $\Theta$ centered at $\xi_\Theta$. The effect of $1_B(D)$ on it can be expressed as:*

$$\check{\varphi}(x) \approx e^{2\pi i \xi_\Theta x} 1_{\mathcal{R}}(x) \xrightarrow{1_B(D)} e^{2\pi i \xi_\Theta x} 1_{\widetilde{\mathcal{R}}}(x),$$

*where $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are arbitrarily translated $\Theta_0^*$ and $\widetilde{\Theta_0^*}$. Recall we partition the $\delta^2$ annulus into $\delta \times \delta^2$ rectangles. Let $\hat{f}$ be the sum of modulated bump functions associated with these rectangles. The effect of $1_B(D)$ on $f$ can be expressed as:*

$$f(x) \approx \sum_{\mathcal{R}} e^{2\pi i \xi_\Theta x} 1_{\mathcal{R}}(x) \xrightarrow{1_B(D)} \sum_{\mathcal{R}} e^{2\pi i \xi_\Theta x} 1_{\widetilde{\mathcal{R}}}(x).$$

*Here $\mathcal{R}$ ranges a set of $\delta^{-1} \times \delta^{-2}$ rectangles with $\delta$-separated directions. In each direction, we can decide the precise postition of the corresponding $\mathcal{R}$.*

# 4 Creating Peaks via Kakeya Type Configuration

By the above intuition, we should arrange the positions of $\mathcal{R}$'s such that:

$$\|\sum_{\mathcal{R}} e^{2\pi i \xi_\Theta x} 1_{\mathcal{R}}\|_p^p \text{ is small,} \tag{3}$$

$$\|\sum_{\mathcal{R}} e^{2\pi i \xi_\Theta x} 1_{\widetilde{\mathcal{R}}}\|_p^p \text{ is large.} \tag{4}$$

Recall $p > 2$. $L^p$ norms for large $p$ is more sensitive to the peaks of a function. Thus $\mathcal{R}$'s should be disjoint to avoid concentration and $\widetilde{\mathcal{R}}$'s should be overlapping to create peaks. Since rectangles are convex, it's impossible to make the entire $(\widetilde{\mathcal{R}} \backslash \mathcal{R})$'s intersect severely. Instead, we pick one side $\mathcal{R}_+$ for each $\widetilde{\mathcal{R}}$ and let $\mathcal{R}_+$'s overlap. See Figure 3. By disjointness, the quantity in (3) is equal to

$$|\bigcup_{\mathcal{R}} \mathcal{R}|.$$

And the quantity in (4) is larger than

$$\int_{\bigcup_{\mathcal{R}} \mathcal{R}_+} |\sum_{\mathcal{R}} e^{2\pi i \xi_\Theta x} 1_{\mathcal{R}_+}|^p.$$

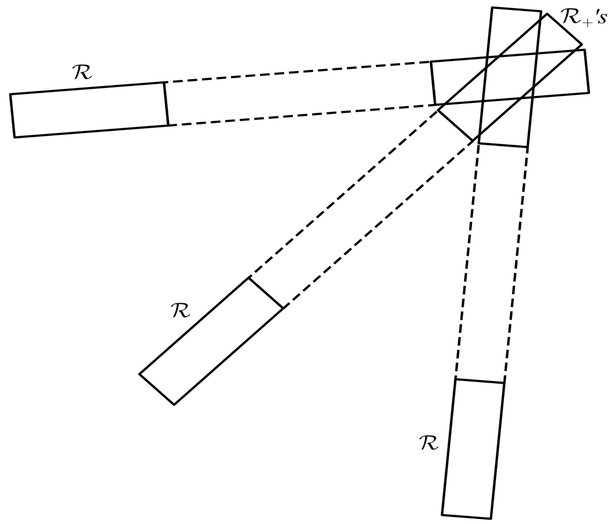


Figure 3: Overlapping

It would be perfect to have

$$|\sum_{\mathcal{R}} e^{2\pi i\xi_\Theta x} 1_{\mathcal{R}_+}| \approx |\sum_{\mathcal{R}} 1_{\mathcal{R}_+}|$$

on $\bigcup_{\mathcal{R}} \mathcal{R}_+$. But these $L^2$ orthogonal waves have oscillation and must cancel a lot under summation. In such situations, we usually expect a square-root cancellation. We should at least be able to achieve:

$$|\sum_{\mathcal{R}} e^{2\pi i\xi_\Theta x} 1_{\mathcal{R}_+}| \approx (\sum_{\mathcal{R}} |e^{2\pi i\xi_\Theta x} 1_{\mathcal{R}_+}|^2)^{\frac{1}{2}} = (\sum_{\mathcal{R}} 1_{\mathcal{R}_+})^{\frac{1}{2}}. \tag{5}$$

Assume $\mathcal{R}_+$'s severely overlap to the extent:

$$|\bigcup_{\mathcal{R}} \mathcal{R}_+| \leq \frac{1}{W} \sum_{\mathcal{R}} |\mathcal{R}_+| \sim \frac{1}{W} |\bigcup_{\mathcal{R}} \mathcal{R}|. \tag{6}$$

Here $W$ is a large number depending on $\delta$. Then we can estimate the quantity in (4) using Hölder's inequality:

$$\begin{aligned}
\int_{\bigcup_{\mathcal{R}} \mathcal{R}_+} |\sum_{\mathcal{R}} e^{2\pi i\xi_\Theta x} 1_{\mathcal{R}_+}|^p &\gtrsim \int_{\bigcup_{\mathcal{R}} \mathcal{R}_+} (\sum_{\mathcal{R}} 1_{\mathcal{R}_+})^{\frac{p}{2}} \\
&\geq (\int_{\bigcup_{\mathcal{R}} \mathcal{R}_+} 1)^{1-\frac{p}{2}} (\int_{\bigcup_{\mathcal{R}} \mathcal{R}_+} \sum_{\mathcal{R}} 1_{\mathcal{R}_+})^{\frac{p}{2}} \\
&\gtrsim W^{\frac{p}{2}-1} |\bigcup_{\mathcal{R}} \mathcal{R}|.
\end{aligned}$$

Here we used $1 - \frac{p}{2} < 0$. Making $W$ arbitrarily large as $\delta \to 0$ will fulfill (3) and (4).

# 5 The Rigorous Construction

Now we make everything rigorous. It suffices to use rectangles with non-negative slopes. Fix $C = 10^{10}$. Given such a rectangle $\mathcal{R}$ of dimension $C\delta^{-1} \times \delta^{-2}$. Its direction is defined to be the unique vector $v \in \mathbb{S}^1 \cap \mathbb{H}_+$ parallel to the long side. And $\mathcal{R}_+ := \mathcal{R} + 10\delta^{-2}v$.

It's simpler to use building blocks which have compact supports in the physical space.

**Proposition 3** (Wave Packets)**.** *Let $\mathcal{R}, \mathcal{R}_+$ as above. There exists a smooth function $f_{\mathcal{R}}$ supported in $\mathcal{R}$ such that $|f_{\mathcal{R}}(x)| \leq 1$ on $\mathcal{R}$ and $|1_B(D)f_{\mathcal{R}}| \sim 1$ on $\frac{1}{3}\mathcal{R}_+$.*

*Proof Sketch.* By translation and rotation, we can assume $\mathcal{R}$ is centered at 0 and $v = (1, 0)$. Take a smooth function $h$ such that $\frac{1}{3}(-\frac{1}{2}, \frac{1}{2})^2 \prec h \prec (-\frac{1}{2}, \frac{1}{2})^2$. Let

$$f_{\mathcal{R}}(x_1, x_2) := e^{2\pi i x_1} h(\delta^2 x_1, \frac{\delta x_2}{C}).$$

There are two steps to prove $|1_B(D)f_{\mathcal{R}}| \sim 1$ on $\frac{1}{3}\mathcal{R}_+$. First, approximate:

$$1_{|\xi|<1}(D)f_{\mathcal{R}} \approx 1_{\xi_1<1}(D)f_{\mathcal{R}}. \tag{7}$$

We use the large constant $C = 10^{10}$ to ensure the difference is really an error. Second, calculate $1_{\xi_1<1}(D)f_{\mathcal{R}}$ explicitly using the Hilbert transform. □

To achieve (5), we put $\pm$ signs in front of $f_{\mathcal{R}}$.

**Proposition 4** (Square-Root Behavior)**.** *Let $p > 2$. Given any family of functions $f_{\mathcal{R}}$ and any $A \subset \mathbb{R}^2$, one can choose $X_{\mathcal{R}} \in \{-1, +1\}$ such that*

$$\int_A |\sum_{\mathcal{R}} X_{\mathcal{R}} f_{\mathcal{R}}|^p \gtrsim \int_A (\sum_{\mathcal{R}} |f_{\mathcal{R}}|^2)^{\frac{p}{2}}.$$

*Proof Sketch.* Let $X_{\mathcal{R}}$ be i.i.d. random variables uniformly distributed on $\{-1, +1\}$. It's not hard to see

$$E(\int_A |\sum_{\mathcal{R}} X_{\mathcal{R}} f_{\mathcal{R}}|^p) \geq \int_A (\sum_{\mathcal{R}} |f_{\mathcal{R}}|^2)^{\frac{p}{2}}.$$

□

And we have seen the rigorous version of (6).

**Proposition 5** (Kakeya Type Configuration [3])**.** *Let $N \in \mathbb{N}$ be a large number of suitable form. Then we can find $N$ disjoint rectangles $\{\mathcal{R}\}$ of dimension $\frac{1}{N} \times 1$ such that*

$$|\bigcup_{\mathcal{R}} \mathcal{R}_+| \lesssim \frac{\log \log N}{\log N} |\bigcup_{\mathcal{R}} \mathcal{R}|.$$

Pick a large $N$. Let $\delta := (CN)^{-1}$ and rescale the rectangles in Proposition 5 into $C\delta^{-1} \times \delta^{-2}$ ones. Then apply Proposition 3 to get $f_{\mathcal{R}}$ for each (rescaled) $\mathcal{R}$. By Proposition 4, we finally obtain

$$\|1_B(D)(\sum_{\mathcal{R}} X_{\mathcal{R}} f_{\mathcal{R}})\|_p \gtrsim (\log N)^{c_p} \|\sum_{\mathcal{R}} X_{\mathcal{R}} f_{\mathcal{R}}\|_p$$

for some choice of $X_{\mathcal{R}} \in \{-1, +1\}$. This is the desired counterexample.

# 6 Remarks

- Let $2 \leq p \leq 4$. One can prove that the square root behavior (5), in the $L^p$ sense, is essentially the best we can achieve [1].
- Proposition 4 holds for general $0 < p < \infty$. To show the general case, one needs to apply Khintchine's inequality. See the reference in [3].
- Fefferman's original argument [2] is slightly different. He used Meyer's lemma, which essentially combines (7) and Proposition 4 together.
- In higher dimensions, the ball multiplier is unbounded for $p \neq 2$ either. In fact, one can prove

  $$\|1_B(D)f\|_p \lesssim \|f\|_p, \forall f \in \mathcal{S}(\mathbb{R}^d) \Longrightarrow \|1_B(D)g\|_p \lesssim \|g\|_p, \forall g \in \mathcal{S}(\mathbb{R}^{d-1}).$$

  See the reference in [2].

# References


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[3] Larry Guth. *Polynomial methods in combinatorics*, volume 64 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2016.